Math 131B-2: Homework 3

Due: April 21, 2014

1. Read sections 4.1-5, 4.8-9 in Apostol.
2. Do problems 3.38, 3.42, 3.17, 3.20, 3.22, 4.7, 4.8, and 4.9 in Apostol.

   Notice that problem 3.38 justifies our assertion that compactness is an absolute, not relative, property. Notice also that 4.8 is a version of the Bolzano-Weierstrass theorem (indeed, it is the most general version we will see).
3. Which of the following subsets of $\mathbb{R}^2$ is compact? Justify your answers.
   - $S_1 = \{(x, y) : 0 \leq xy \leq 1\}$.
   - $S_2 = \{(n, \frac{1}{n}) : n \in \mathbb{N}\}$.
   - $S_3 = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq x^2\}$.
   - $S_4 = \{(x, y) : x^2 + y^2 = 9\}$.
   - $S_5 = \{(x, y) : \frac{x^2}{4} + \frac{y^2}{25} < 1\}$.
4. *Two metrics with the same convergence properties.* Consider the space $\mathbb{R}^n$ with two metrics $||\cdot||$ and d_2 defined as follows. If $\mathbf{x} = (x_1, \cdots x_n)$ and $\mathbf{y} = (y_1, \cdots y_n)$, we let

$$||\mathbf{x} - \mathbf{y}|| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

$$d_2(\mathbf{x}, \mathbf{y}) = \max\{|x_i - y_i| : 1 \leq i \leq n\}$$

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   That is, $||\cdot||$ is the usual metric and d_2 is the "box metric."

   - Let $\mathbf{x} \in \mathbb{R}^n$. Show that any ball $B_{(\mathbb{R}^n, ||\cdot||)}(\mathbf{x}; r)$ contains a ball $B_{(\mathbb{R}^n, d_2)}(\mathbf{x}; r')$ for some $r' \leq r$. Likewise show that any ball $B_{(\mathbb{R}^n, d_2)}(\mathbf{x}; r)$ contains a ball $B_{(\mathbb{R}^n, ||\cdot||)}(\mathbf{x}; r')$ for some $r' \leq r$. The inequalities you proved on Homework 1 may be useful.
   - Show that $U \subset \mathbb{R}^n$ is open in $(\mathbb{R}^n, ||\cdot||)$ if and only if it is an open in $(\mathbb{R}^n, d_2)$.
   - Show that a sequence $\{\mathbf{x}^k\} = \{x_1^k, \cdots x_n^k\}$ converges to some $\mathbf{x}^0$ in $(\mathbb{R}^n, ||\cdot||)$ if and only if it converges to $\mathbf{x}^0$ in $(\mathbb{R}^n, d_2)$. (Here superscripts denote place in the sequence and subscripts denote the Cartesian coordinates of a point.)
   - Show that $\{\mathbf{x}^k\}$ is a Cauchy sequence in $(\mathbb{R}^n, ||\cdot||)$ if and only if $\{\mathbf{x}_n\}$ is a Cauchy sequence in $(\mathbb{R}^n, d_2)$. Conclude that $(\mathbb{R}^n, d_2)$ is complete.